ON RANGE IDEMPOTENT MATRICES

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Abstract:

In this paper, the concept of range idempotent matrix is introduced as an analogue of range symmetric matrices over the complex field and its basic properties are discussed.

Keywords:

Range hermitian matrix, range idempotent matrix.

I. INTRODUCTION

Throughout we shall deal with $\mathbb{C}^{n\times n}$, the space of $n\times n$ complex matrices. Let $\mathbb{C}^n$ be the space of all complex n - tuples. For any $A\in\mathbb{C}^{n\times n}$, $\bar{A}$, A^* denote the conjugate and conjugate transpose of A respectively. The Range space of $A\in\mathbb{C}^{n\times n}$ is a subspace of $\mathbb{C}^n$ is denoted by $\mathcal{R}(A)$ and is defined by $\mathcal{R}(A)=\{\,y\in\mathbb{C}^n : y=Ax\ for\ some\ x\in\mathbb{C}^n\}$. For any $A,B\in\mathbb{C}^{n\times n}$, $\mathcal{R}(AB)\subseteq\mathcal{R}(A)$ [1]. The Null space of $A\in\mathbb{C}^{n\times n}$ is a subspace of $\mathbb{C}^n$ is denoted by $\mathcal{N}(A)$ and is defined by $\mathcal{N}(A)=\{\,x\in\mathbb{C}^n : Ax=0\}$. As a generalization of normality, the concept of EP matrices over the complex field was introduced by Schwerdtfeger [2]. Any matrix $A\in\mathbb{C}^{n\times n}$ is said to be EP or range hermitian if $\mathcal{R}(A)=\mathcal{R}(A^*)$ or $\mathcal{N}(A)=\mathcal{N}(A^*)$ [1]. Any matrix $A\in\mathbb{C}^{n\times n}$ is said to be idempotent if $A^2=A$. Here, we developed the concept of idempotent matrices to range idempotent matrices as a similar extension of hermitian to range hermitian matrices.

II. RANGE IDEMPOTENT MATRICES

In this section we present the definition of range idempotent matrix and investigated some of its basic properties.

Definition:2.1

Any matrix $A\in\mathbb{C}^{n\times n}$ is said to be range idempotent if $\mathcal{R}(A^2)=\mathcal{R}(A)$.

Example:2.2

$A=\begin{bmatrix} i & i\\ i & i\end{bmatrix}\in\mathbb{C}^{2\times 2}$ is a range idempotent matrix, since $\mathcal{R}(A^2)=\mathcal{R}(A)$.

Remark:2.3

Every idempotent matrix is always a range idempotent. But the converse need not be true, which can be seen from the following example.

Example:2.4

Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \neq A$. But $\mathcal{R}(A^2) = \mathcal{R}(A)$. Thus A is range idempotent but not idempotent.

Theorem:2.5

Let $A \in \mathbb{C}^{n\times n}$ be range idempotent matrix, then $\mathcal{R}(A) = \mathcal{R}(A^k), k > 2$.

Proof:

Since for $A, B \in \mathbb{C}^{n\times n}$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, we have $\mathcal{R}(A^k) \subseteq \mathcal{R}(A)$

Conversely, we have to prove $\mathcal{R}(A) \subseteq \mathcal{R}(A^k)$

For $y \in \mathcal{R}(A) \Longrightarrow \quad Ax = y$

$\Longrightarrow \quad A^2x_1 = y$ $\quad$ ($\because A$ is range idempotent)

$\Longrightarrow \quad AA^2x_2 = y$ $\quad$ ($\because A$ is range idempotent)

$\Longrightarrow \quad A^2Ax_2 = y$

$\vdots$

$\Longrightarrow \quad A^{k-1}Ax_{k-1} = y$

$\Longrightarrow \quad A^kx_{k-1} = y$

$\Longrightarrow \quad y \in \mathcal{R}(A^k)$.

Therefore, $\mathcal{R}(A) \subseteq \mathcal{R}(A^k)$

Hence $\mathcal{R}(A) = \mathcal{R}(A^k)$.

Theorem:2.6

Let $A \in \mathbb{C}^{n\times n}$ be range idempotent matrix iff $\bar{A}$ is range idempotent.

Proof:

A is range idempotent matrix $\Leftrightarrow \mathcal{R}(A^2) = \mathcal{R}(A) \quad \Leftrightarrow \quad \overline{\mathcal{R}(A^2)} = \overline{\mathcal{R}(A)}$

$\Leftrightarrow \mathcal{R}(\bar{A}) = \mathcal{R}(\overline{A^2}) \quad \Leftrightarrow \quad \mathcal{R}(\bar{A}) = \mathcal{R}(\bar{A}^2)$

$\Leftrightarrow \bar{A}$ is range idempotent.

Theorem:2.7

Let $A, B \in \mathbb{C}^{n\times n}$ be range idempotent matrices with $AB = BA$. Then $\mathcal{R}(AB)^2 = \mathcal{R}(BA)$.

Proof:

Since for $A, B \in \mathbb{C}^{n\times n}$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, we have $\mathcal{R}(AB)^2 \subseteq \mathcal{R}(AB)$

Conversely, we have to prove $\mathcal{R}(AB) \subseteq \mathcal{R}(AB)^2$

Let $y \in \mathcal{R}(AB) \quad \Longrightarrow \quad ABx = y$

$\Longrightarrow \quad AB^2x_1 = y$ $\quad$ ($\because B$ is range idempotent)

$\Longrightarrow \quad B^2Ax_1 = y$ $\quad$ ($\because AB = BA$)

$\Longrightarrow \quad B^2A^2x_2 = y$ $\quad$ ($\because A$ is range idempotent)

$\Longrightarrow \quad (AB)^2x_2 = y$ $\quad$ ($\because AB = BA$)

$\Longrightarrow \quad y \in \mathcal{R}(AB)^2$.

Therefore, $\mathcal{R}(AB) \subseteq \mathcal{R}(AB)^2$. Hence $\mathcal{R}(AB)^2 = \mathcal{R}(BA)$.

Theorem:2.8

Let $A_1, A_2 \dots A_k \in \mathbb{C}^{n\times n}$ be range idempotent and commutative matrices. Then $\mathcal{R}(A_1.A_2 \dots A_k)^2 = \mathcal{R}(A_1.A_2 \dots A_k)$.

Proof:

Since for $A, B \in \mathbb{C}^{n\times n}$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$.

We have $\mathcal{R}(A_1.A_2 \dots A_k)^2 \subseteq \mathcal{R}(A_1.A_2 \dots A_k)$.

Conversely, we have to prove $\mathcal{R}(A_1.A_2 \dots A_k) \subseteq \mathcal{R}(A_1.A_2 \dots A_k)^2$.

Let $y \in \mathcal{R}(A_1.A_2 \dots A_k)$

$$
\begin{aligned}
&\Longrightarrow\ A_1.A_2 \dots A_k x = y \\
&\Longrightarrow\ A_1.A_2 \dots A_k^2 x_1 = y && (\because A_k \text{ is range idempotent}) \\
&\Longrightarrow\ A_1.A_2 \dots A_k^2 A_{k-1} x_1 = y && (\because A_i\text{'s are commutative}) \\
&\qquad\vdots \\
&\Longrightarrow\ A_k^2 \dots A_2^2 . A_1^2 x_k = y \\
&\Longrightarrow\ (A_1.A_2 \dots A_k)^2 x_k = y \\
&\Longrightarrow\ y \in \mathcal{R}(A_1.A_2 \dots A_k)^2.
\end{aligned}
$$

Therefore, $\mathcal{R}(A_1.A_2 \dots A_k) \subseteq \mathcal{R}(A_1.A_2 \dots A_k)^2$.

Hence $\mathcal{R}(A_1.A_2 \dots A_k)^2 = \mathcal{R}(A_1.A_2 \dots A_k)$.

Theorem:2.9

Let $A \in \mathbb{C}^{n\times n}$ be range idempotent and EP matrices. Then

1. $\mathcal{R}(A) = \mathcal{R}(AA^*)$
2. $\mathcal{R}(A) = \mathcal{R}(A^k A^*)$

Proof:

$$
\begin{aligned}
\text{Let } y \in \mathcal{R}(A) &\Longleftrightarrow\ Ax = y \\
&\Longleftrightarrow\ AAx_1 = y && (\because A \text{ is range idempotent}) \\
&\Longleftrightarrow\ AA^* x_2 = y && (\because A \text{ is EP}) \\
&\Longleftrightarrow\ y \in \mathcal{R}(AA^*).
\end{aligned}
$$

Hence $\mathcal{R}(A) = \mathcal{R}(AA^*)$.

Similarly,

$$
\begin{aligned}
\text{Let } y \in \mathcal{R}(A) &\Longleftrightarrow\ Ax = y \\
&\Longleftrightarrow\ AAx_1 = y && (\because A \text{ is range idempotent}) \\
&\Longleftrightarrow\ AA^2 x_2 = y && (\because A \text{ is range idempotent}) \\
&\qquad\vdots \\
&\Longleftrightarrow\ A^k A x_k = y \\
&\Longleftrightarrow\ A^k A^* x_{k+1} = y && (\because A \text{ is EP}) \\
&\Longleftrightarrow\ y \in \mathcal{R}(A^k A^*).
\end{aligned}
$$

Hence $\mathcal{R}(A) = \mathcal{R}(A^k A^*)$.

Theorem:2.10

Let $A \in \mathbb{C}^{n\times n}$ be range idempotent and EP matrices iff $\mathcal{N}(A) = \mathcal{N}(A^2)$.

Proof:

$$\begin{aligned}
\text{Let } \mathcal{R}(A) = \mathcal{R}(A^2) \quad &\Leftrightarrow \quad \mathcal{N}(A^*)^{\perp} = \mathcal{N}((A^2)^*)^{\perp} && (\because\ \mathcal{R}(A) = \mathcal{N}(A^*)^{\perp})\\
&\Leftrightarrow \quad \mathcal{N}(A^*) = \mathcal{N}((A^2)^*)\\
&\Leftrightarrow \quad \mathcal{N}(A^*) = \mathcal{N}((A^*)^2)\\
&\Leftrightarrow \quad \mathcal{N}(A) = \mathcal{N}(A^2). && (\because\ EP\ matrix)
\end{aligned}$$

Theorem:2.11

Let $A \in \mathbb{C}^{n\times n}$ be range idempotent and EP matrices. Then

1. $\mathcal{R}(AA^*A) = \mathcal{R}(AA^*)^2$
2. $\mathcal{R}(A^kA^*A) = \mathcal{R}(A^kA^*)^2$

Proof:

$$\begin{aligned}
\text{Let } y \in \mathcal{R}(AA^*A) \quad &\Leftrightarrow \quad AA^*Ax = y\\
&\Leftrightarrow \quad AA^*A^2x_1 = y && (\because A \text{ is range idempotent})\\
&\Leftrightarrow \quad AA^*AAx_1 = y\\
&\Leftrightarrow \quad AA^*AA^*x_2 = y && (\because A \text{ is EP})\\
&\Leftrightarrow \quad (AA^*)^2x_2 = y\\
&\Leftrightarrow \quad y \in \mathcal{R}(AA^*)^2.
\end{aligned}$$

Hence $\mathcal{R}(AA^*A) = \mathcal{R}(AA^*)^2$.

Similarly,

$$\begin{aligned}
\text{Let } y \in \mathcal{R}(A^kA^*A) \quad &\Leftrightarrow \quad A^kA^*Ax = y\\
&\Leftrightarrow \quad A^kA^*A^2x_1 = y && (\because A \text{ is range idempotent})\\
&\Leftrightarrow \quad A^kA^*AAx_1 = y\\
&\Leftrightarrow \quad A^kA^*AA^2x_2 = y\\
&\Leftrightarrow \quad A^kA^*A^2Ax_2 = y\\
&\Leftrightarrow \quad A^kA^*A^2A^2x_3 = y\\
&\Leftrightarrow \quad A^kA^*A^3Ax_3 = y\\
&\qquad\quad \vdots\\
&\Leftrightarrow \quad A^kA^*A^kAx_k = y\\
&\Leftrightarrow \quad A^kA^*A^kA^*x_{k+1} = y && (\because A \text{ is EP})\\
&\Leftrightarrow \quad (A^kA^*)^2x_{k+1} = y\\
&\Leftrightarrow \quad y \in \mathcal{R}(A^kA^*)^2.
\end{aligned}$$

Hence $\mathcal{R}(A^kA^*A) = \mathcal{R}(A^kA^*)^2$.

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